

# Real Root Isolation of Polynomial Equations Based on Hybrid Computation

Shen Fei<sup>1</sup>, Wu Wenyuan<sup>2</sup>, and Xia Bican<sup>1</sup>

<sup>1</sup> LMAM & School of Mathematical Sciences, Peking University

<sup>2</sup> Chongqing Institute of Green and Intelligent Technology, Chinese Academy of Sciences

shenfei@pku.edu.cn    wuwenyuan@cigit.ac.cn    xbc@math.pku.edu.cn

**Abstract.** A new algorithm for real root isolation of polynomial equations based on hybrid computation is presented in this paper. Firstly, the approximate (complex) zeros of the given polynomial equations are obtained via homotopy continuation method. Then, for each approximate zero, an initial box relying on Kantorovich theorem is constructed, which contains the corresponding accurate zero. Finally, Krawczyk interval iteration with interval arithmetic is applied to the initial boxes so as to check whether or not the corresponding approximate zeros are real and to obtain the real root isolation boxes. Meanwhile, an empirical construction of initial box is provided for higher performance. And our experiments on many benchmarks show that the new hybrid method is more efficient, compared with the traditional symbolic approaches.

## 1 Introduction

The *Real Roots Isolation* of polynomial equations is a procedure that uses disjoint regions to isolate all the distinct real roots of polynomial equations, with only one root in each region. Formally speaking, let  $\mathbf{F} = (f_1, f_2, \dots, f_n)^T$  be polynomial equations defined on  $\mathbf{R}^n$ , i.e.  $f_i \in \mathbf{R}[x_1, x_2, \dots, x_n]$ . Suppose  $\mathbf{F}(\mathbf{x}) = 0$  has only finite many real roots, say  $\xi^{(1)}, \xi^{(2)}, \dots, \xi^{(m)}$ . The target of real root isolation is to compute a family of regions  $S_1, S_2, \dots, S_m$ ,  $S_j \subset \mathbf{R}^n$  ( $1 \leq j \leq m$ ), such that  $\xi^{(j)} \in S_j$  and  $S_i \cap S_j = \emptyset$  ( $1 \leq i, j \leq m$ ). Usually, we use rectangular boxes to denote the regions above. So we often call these isolated boxes *intervals* in this paper.

Real root isolation is an important problem in symbolic computation. It can be viewed as a kind of exact algorithm for solving equations since no root formula is available in general situation. It is also a critical part of some other important algorithms, such as CAD and real root classification for semi-algebraic system, etc. Improvement on real root isolation will benefit all of these algorithms.

We impose some hypothesis on the problem discussed here. First is that the system is square, i.e. the number of equations is the same as that of variables. Then we only handle the systems with finite many roots. Positive dimensional solution is beyond the scope of this paper. Moreover, we suppose that the Jacobian matrix of  $\mathbf{F}$  is non-singular at each root of  $\mathbf{F}(\mathbf{x}) = 0$ . So we only deal with

the simple root cases. For the singular situation, the deflation method [9,10,21,7] can be applied, which is one of our ongoing work.

Most of the previous real root isolation algorithms are based on symbolic computations. For instance, the Uspensky algorithm [6] based on Descartes' rule is for polynomials in one variable. In multi-variable scenario, we have "First algorithm" based on monotonicity [24] and "Second algorithm" based on "upper-lower bound" polynomial [23]. There are also some other algorithms based on different techniques, see for example [5,3,13,4].

An advantage of those symbolic methods is that exact results can be obtained since they use symbolic computation and some of them can be extended to semi-algebraic systems. However, there are also some disadvantages. Some of these method could only handle the isolation of complex zeros. And some of them need to triangularize the system first, which is unacceptable in computation when the system is complicated sometimes, such as more variables or high degrees. While some methods that do not use triangularization have to give a huge initial interval to include all the real roots [25,26], which is extremely inefficient.

In order to avoid these problems and design a new algorithm that could efficiently solve more complicated systems and provide accurate interval results, we employ hybrid computation to take both the advantages of symbolic and numerical methods.

The basic idea of this paper is to use numerical method to obtain all the approximate zeros of polynomial systems, including possible non-real ones. With these approximations, small initial intervals which contains the corresponding real roots are constructed. We then apply symbolic method to these initial intervals to verify whether there is a real root in it or not. The main method we use in numerical computation is homotopy continuation, and for symbolic process we use Krawczyk iteration.

Most of the work in this paper comes from [16]. In section 2, we will introduce some preliminaries, including homotopy continuation and interval arithmetic. A new real root isolation algorithm is discussed in section 3. To test our new method, our experimental results on benchmarks together with comparison and analysis will be presented in section 4. Finally, there is a summary in section 5 and some future work will also be discussed.

## 2 Preliminary

We introduce in this section some basic theories and tools that would be used in our algorithm.

### 2.1 Homotopy Continuation Method

Homotopy continuation method is an important numerical computation method, which is used in various fields. We only treat it as an "algorithm black box" here, where the input is a polynomial system, and the output is its approximate zeros. Please find the details about the theory in [11,17].

For our purpose, it is convenient to apply some existing softwares, such as Hom4ps-2.0 [12], PHCpack [18] and HomLab [19].

In our implementation, we use Hom4ps-2.0, which could return all the approximate complex zeros of a given polynomial system efficiently, along with residues and condition numbers.

## 2.2 Interval arithmetic

Interval arithmetic plays an important role in real root isolation algorithms [25,26,23]. The two main differences between our new algorithm and the traditional ones in [25,26] are: 1) Verification only carry out on the localized “small” intervals; 2) symbolic computation is replaced with floating point numerical computation.

Most of the interval operations in this paper’s algorithms are based on Rump’s floating point verification work [15] and accomplished by using the Matlab package Intlab [14], including interval arithmetic operations and Jacobian matrix, Hessian matrix calculations.<sup>1</sup> So we will focus on the real root isolation algorithm.

**Basic concepts** We introduce some basic interval arithmetic theories in this section. See reference [20] for more details.

For given numbers  $\underline{x}, \bar{x} \in \mathbf{R}$ , if  $\underline{x} \leq \bar{x}$ , then we call

$$X = [\underline{x}, \bar{x}] = \{x \in \mathbf{R} | \underline{x} \leq x \leq \bar{x}\}$$

a *bounded close interval*, or *interval* for short. Denote by  $I(\mathbf{R})$  the set of all the bounded close intervals on  $\mathbf{R}$ , and  $I(A) = \{X \in I(\mathbf{R}) | X \subseteq A\}$  all the intervals on  $A \subseteq \mathbf{R}$ . Especially, if  $\underline{x} = \bar{x}$ , we call  $X$  a *point interval*.

For intervals, there are some common quantities:

**midpoint**  $\text{mid}(X) = (\underline{x} + \bar{x})/2$

**width**  $W(X) = \bar{x} - \underline{x}$

**radius**  $\text{rad}(X) = \frac{1}{2}W(X)$

**low end point**  $\inf(X) = \underline{x}$

**high end point**  $\sup(X) = \bar{x}$

Obviously we have  $X = [\text{mid}(X) - \text{rad}(X), \text{mid}(X) + \text{rad}(X)]$ . An interval is usually expressed by its midpoint and radius. For example, if  $m = \text{mid}(X)$ ,  $r = \text{rad}(X)$ , then we can write the formula above as  $X = \text{midrad}(m, r)$ .

We can also define the arithmetic operations over intervals. Let  $X = [\underline{x}, \bar{x}]$ ,  $Y = [\underline{y}, \bar{y}] \in I(\mathbf{R})$ ,

$$\begin{aligned} - X + Y &= [\underline{x} + \underline{y}, \bar{x} + \bar{y}] \\ - X - Y &= [\underline{x} - \bar{y}, \bar{x} - \underline{y}] \\ - X \cdot Y &= [\min(\underline{x}\underline{y}, \underline{x}\bar{y}, \bar{x}\underline{y}, \bar{x}\bar{y}), \max(\underline{x}\underline{y}, \underline{x}\bar{y}, \bar{x}\underline{y}, \bar{x}\bar{y})] \end{aligned}$$

---

<sup>1</sup> See reference [15], Section 11, *Automatic differentiation*.

$$- X/Y = [\underline{x}, \bar{x}] \cdot [1/\bar{y}, 1/\underline{y}], \quad 0 \notin Y$$

A vector is called an *interval vector* if all its components are intervals. *Interval matrix* can be similarly defined. For interval vectors and interval matrices, the concepts such as midpoint, width, radius, etc, and the arithmetic operations are defined in components.

Let  $f : \mathbf{R}^n \rightarrow \mathbf{R}$  be a function, if there exists an interval map

$$F : I(\mathbf{R}^n) \rightarrow I(\mathbf{R})$$

such that for all  $x_i \in X_i (i = 1, 2, \dots, n)$ ,

$$F([x_1, x_1], [x_2, x_2], \dots, [x_n, x_n]) = f(x_1, x_2, \dots, x_n)$$

holds, then we call  $F$  an *interval expand* of  $f$ .

We call  $F : I(\mathbf{R}^n) \rightarrow I(\mathbf{R})$  an interval map with *inclusive monotonicity* if  $\mathbf{X} \subseteq \mathbf{Y}$  implies  $F(\mathbf{X}) \subseteq F(\mathbf{Y})$  for any given intervals  $\mathbf{X}$  and  $\mathbf{Y}$ . The definitions above can all be extended to the situations in  $I(\mathbf{R}^n) \rightarrow I(\mathbf{R}^n)$ . And it is easy to prove that all the polynomial operations satisfy the inclusive monotonicity.

**Krawczyk operator** Krawczyk operator plays a key role in the real root verification of interval arithmetic. The main accomplishment comes from the work of Krawczyk and Moore. We only list some important results here. Complete proofs can be found in [20].

Suppose  $\mathbf{f} : D \subseteq \mathbf{R}^n \rightarrow \mathbf{R}^n$  is continuous differentiable on  $D$ . Consider the equation

$$\mathbf{f}(\mathbf{x}) = 0 \tag{1}$$

Let  $\mathbf{f}'$  be the Jacobi matrix of  $\mathbf{f}$ ,  $\mathbf{F}$  and  $\mathbf{F}'$  be the interval expand of  $\mathbf{f}$  and  $\mathbf{f}'$  with inclusive monotonicity, respectively. For  $\mathbf{X} \in I(D)$  and any  $\mathbf{y} \in \mathbf{X}$ , define *Krawczyk operator* as:

$$K(\mathbf{y}, \mathbf{X}) = \mathbf{y} - Y\mathbf{f}(\mathbf{y}) + (I - Y\mathbf{F}'(\mathbf{X}))(\mathbf{X} - \mathbf{y}) \tag{2}$$

where  $Y$  is any  $n \times n$  non-singular matrix.

Especially, we assign  $\mathbf{y} = \text{mid}(\mathbf{X})$ , so Formula (2) becomes

$$K(\mathbf{X}) = \text{mid}(\mathbf{X}) - Y\mathbf{f}(\text{mid}(\mathbf{X})) + (I - Y\mathbf{F}'(\mathbf{X}))\text{rad}(\mathbf{X})[-1, 1]. \tag{3}$$

Formula (3) is often used in practise.

The reason why Krawczyk operator is so important is that it has some nice properties.

**Proposition 1** Suppose  $K(\mathbf{y}, \mathbf{X})$  is defined as Formula (2), then

1. If  $\mathbf{x}^* \in \mathbf{X}$  is a root of Equation (1), then for any  $\mathbf{y} \in \mathbf{X}$ , we have  $\mathbf{x}^* \in K(\mathbf{y}, \mathbf{X})$ ;
2. For any  $\mathbf{y} \in \mathbf{X}$ , if  $\mathbf{X} \cap K(\mathbf{y}, \mathbf{X}) = \emptyset$  holds, then there is no roots in  $\mathbf{X}$ ;

3. For any  $\mathbf{y} \in \mathbf{X}$  and any non-singular matrix  $Y$ , if  $K(\mathbf{y}, \mathbf{X}) \subseteq \mathbf{X}$  holds, then Equation (1) has a solution in  $\mathbf{X}$ ;
4. Moreover, for any  $\mathbf{y} \in \mathbf{X}$  and any non-singular matrix  $Y$ , if  $K(\mathbf{y}, \mathbf{X})$  is strict inclusive in  $\mathbf{X}$ , then Equation (1) has only one root in  $\mathbf{X}$ .

With the properties above, we can easily develop a real root verification method, which is a little different from the classical one, and will be explained later in this paper.

Meanwhile, with the hypothesis we set in introduction, all the systems considered here are non-singular ones with only simple roots. So the Jacobian matrix at the zeros are all invertible. Thus, we often set  $Y = (\text{mid}\mathbf{F}'(\mathbf{X}))^{-1}$  and the Krawczyk operator becomes

$$\begin{aligned} K(\mathbf{X}) = & \text{mid}(\mathbf{X}) - (\text{mid}\mathbf{F}'(\mathbf{X}))^{-1} \mathbf{f}(\text{mid}(\mathbf{X})) \\ & + (I - (\text{mid}\mathbf{F}'(\mathbf{X}))^{-1} \mathbf{F}'(\mathbf{X})) \text{rad}(\mathbf{X})[-1, 1]. \end{aligned} \quad (4)$$

This is also called the Moore form of Krawczyk operator.

### 3 Real root isolation algorithm

In this section, we will present our new algorithm for real root isolation based on hybrid computation. As mentioned before, our idea is to construct the initial intervals corresponding to the approximate zeros obtained by homotopy continuation, then verify them via Krawczyk interval iteration to obtain the isolation results. In the end, we combine these sub-procedures to give the final algorithm description.

#### 3.1 Construction of initial intervals

To apply Krawczyk interval iteration, obviously the construction of initial intervals is a key procedure. We should guarantee both the *correctness* and *efficiency*, that is, make sure the initial box contains the corresponding accurate real root, and keep the interval radius as small as possible so as to shorten the iteration time.

Thus a valid error estimation for the initial approximate zeros should be established. And we discuss this issue in both theory and practice aspects here.

**Error estimation theory** The core problem of the construction of initial box is the choice of interval radius, which is indeed an error estimation for the approximate zero. There are dozen of error analysis for this question, from classic results to modern ones, especially about the Newton method. For example, in [2], S. Smale et al. gave a detailed analysis. However, their method requires computation of high order derivatives, which is not so convenient for our problem.

Here we employ Kantorovich Theorem to give our error estimation.

**Theorem 2 (Kantorovich)** *Let  $X$  and  $Y$  be Banach spaces,  $F : D \subseteq X \rightarrow Y$  be an operator, which is Fréchet differentiable on an open convex set  $D_0 \subseteq D$ . For equation  $F(x) = 0$ , if the given approximate zero  $x_0 \in D_0$  meets the following three conditions:*

1.  $F'(x_0)^{-1}$  exists, and there are real numbers  $B$  and  $\eta$  such that

$$\|F'(x_0)^{-1}\| \leq B, \quad \|F'(x_0)^{-1}F(x_0)\| \leq \eta,$$

2.  $F$  satisfies the Lipschitz condition on  $D_0$ :

$$\|F'(x) - F'(y)\| \leq K\|x - y\|, \quad \forall x, y \in D_0,$$

3.  $h = BK\eta \leq \frac{1}{2}$ ,  $O(x_0, \frac{1-\sqrt{1-2h}}{h}\eta) \subset D_0$ ,

then we claim that:

1.  $F(x) = 0$  has a root  $x^*$  in  $\overline{O(x_0, \frac{1-\sqrt{1-2h}}{h}\eta)} \subset D_0$ , and the sequence  $\{x_k : x_{k+1} = x_k - F'(x_k)^{-1}F(x_k)\}$  of Newton method converges to  $x^*$ ;
2. For the convergence of  $x^*$ , we have:

$$\|x^* - x_{k+1}\| \leq \frac{\theta^{2^{k+1}}(1 - \theta^2)}{\theta(1 - \theta^{2^{k+1}})}\eta \quad (5)$$

$$\text{where } \theta = \frac{1-\sqrt{1-2h}}{1+\sqrt{1-2h}};$$

3. The root  $x^*$  is unique in  $D_0 \cap \overline{O(x_0, \frac{1+\sqrt{1-2h}}{h}\eta)}$ .

In the theorem,  $\overline{O(x, r)}$  denotes the ball neighborhood whose center is  $x$  and radius is  $r$ , and  $O(x, r)$  refers to the closure of the ball neighborhood. The proof can be found in [8].

Since the approximation  $x_0$  is already the result of homotopy process, what we care about is the initial interval w.r.t.  $x_0$ , i.e. the proper upper bound for  $\|x^* - x_0\|$ . So we have the following proposition, which is a direct corollary of Kantorovich Theorem.

**Proposition 3** *Let  $\mathbf{F} = (f_1, f_2, \dots, f_n)^T$  be a polynomial system, where  $f_i \in \mathbf{R}[x_1, x_2, \dots, x_n]$ . Denote by  $\mathbf{J}$  the Jacobian matrix of  $\mathbf{F}$ . For an approximation  $\mathbf{x}_0 \in \mathbf{R}^n$ , if the following conditions hold:*

1.  $\mathbf{J}^{-1}(\mathbf{x}_0)$  exists, and there are real numbers  $B$  and  $\eta$  such that

$$\|\mathbf{J}^{-1}(\mathbf{x}_0)\| \leq B, \quad \|\mathbf{J}^{-1}(\mathbf{x}_0)\mathbf{F}(\mathbf{x}_0)\| \leq \eta,$$

2. There exists a ball neighbourhood  $O(\mathbf{x}_0, \omega)$  such that  $\mathbf{J}(\mathbf{x})$  satisfies the Lipschitz condition on it:

$$\|\mathbf{J}(\mathbf{x}) - \mathbf{J}(\mathbf{y})\| \leq K\|\mathbf{x} - \mathbf{y}\|, \quad \forall \mathbf{x}, \mathbf{y} \in O(\mathbf{x}_0, \omega)$$

3. Let  $h = BK\eta$ ,

$$h \leq \frac{1}{2}, \text{ and } \omega \geq \frac{1 - \sqrt{1 - 2h}}{h}\eta,$$

then  $\mathbf{F}(\mathbf{x}) = 0$  has only one root  $\mathbf{x}^*$  in  $\overline{O(\mathbf{x}_0, \omega)}$ .

*Proof.* We consider  $F$  as an operator on  $\mathbf{R}^n \rightarrow \mathbf{R}^n$ , obviously it is Fréchet differentiable, and from

$$\mathbf{F}(\mathbf{x} + \mathbf{h}) = \mathbf{F}(\mathbf{x}) + \mathbf{J}(\mathbf{x})\mathbf{h} + o(\mathbf{h})$$

we can get

$$\lim_{\mathbf{h} \rightarrow 0} \frac{\|\mathbf{F}(\mathbf{x} + \mathbf{h}) - \mathbf{F}(\mathbf{x}) - \mathbf{J}(\mathbf{x})\mathbf{h}\|}{\|\mathbf{h}\|} = 0.$$

Thus the first order Fréchet derivative of  $\mathbf{F}$  is just the Jacobian matrix  $\mathbf{J}$ , i.e.  $\mathbf{F}'(\mathbf{x}) = \mathbf{J}(\mathbf{x})$ . So by Theorem 2, the proof is completed immediately after checking the situation of  $\|\mathbf{x}^* - \mathbf{x}_0\|$ .

It is easy to know that  $\frac{1 - \sqrt{1 - 2h}}{h} \leq 2$ . So we can just assign  $\omega = 2\eta$ . Then we need to check whether  $K \leq \frac{1}{2B\eta}$  in the neighborhood  $O(\mathbf{x}_0, 2\eta)$ . Even though the initial  $\mathbf{x}_0$  does not satisfy the conditions, we can still find a proper  $\mathbf{x}_k$  after several Newton iterations, since  $\eta$  will approach zero,  $B$  is fixed and we only need to find an upper bound for the Lipschitz constant  $K$ .

**Constructive algorithm** Now we will give a constructive procedure for the Lipschitz constant  $K$ .

Let  $J_{ij} = \partial f_i / \partial x_j$ , apply mean value theorem to each element of  $\mathbf{J}$  on  $O(\mathbf{x}_0, \omega)$  to get

$$J_{ij}(\mathbf{y}) = J_{ij}(\mathbf{x}) + \nabla J_{ij}(\zeta_{ij}) \cdot (\mathbf{y} - \mathbf{x}), \quad \forall \mathbf{x}, \mathbf{y} \in O(\mathbf{x}_0, \omega) \quad (6)$$

where  $\nabla = (\partial/\partial x_1, \partial/\partial x_2, \dots, \partial/\partial x_n)$  is the gradient operator, and  $\zeta_{ij}$  is some point between  $\mathbf{x}$  and  $\mathbf{y}$ , i.e.  $\zeta_{ij} = \mathbf{x} + \kappa_{ij}(\mathbf{y} - \mathbf{x})$  for some  $\kappa_{ij} \in [0, 1]$ . Setting  $(\nabla J_{ij}(\zeta_{ij}) \cdot (\mathbf{y} - \mathbf{x}))_{n \times n} = \Delta \mathbf{J}$ , we have

$$\begin{aligned} \|\Delta \mathbf{J}\|_\infty &= \|(|\nabla J_{ij}(\zeta_{ij}) \cdot (\mathbf{y} - \mathbf{x})|)_{n \times n}\|_\infty \\ &\leq \|(\|\nabla J_{ij}(\zeta_{ij})\|_2 \|\mathbf{y} - \mathbf{x}\|_2)_{n \times n}\|_\infty \\ &\leq n \|(\|\nabla J_{ij}(\zeta_{ij})\|_\infty)_{n \times n}\|_\infty \cdot \|\mathbf{y} - \mathbf{x}\|_\infty \\ &= n \max_{1 \leq i \leq n} \sum_{j=1}^n \|\nabla J_{ij}(\zeta_{ij})\|_\infty \cdot \|\mathbf{y} - \mathbf{x}\|_\infty. \end{aligned} \quad (7)$$

Note that  $\nabla J_{ij}(\zeta_{ij})$  is a vector, so if we use  $|\cdot|_{\max}$  to denote the maximum module component of a vector, then we have

$$\|\Delta \mathbf{J}\|_\infty \leq n \cdot \max_{1 \leq i \leq n} \sum_{j=1}^n |\nabla J_{ij}(\zeta_{ij})|_{\max} \cdot \|\mathbf{y} - \mathbf{x}\|_\infty. \quad (8)$$

Let  $H_i = (\frac{\partial^2 f_i}{\partial x_j \partial x_k})_{n \times n}$  be the Hessian matrix of  $f_i$ , and let  $H_i = (h_1^{(i)}, \dots, h_n^{(i)})$ , where  $h_j^{(i)}$  are the column vectors. Then we have

$$\|\Delta \mathbf{J}\|_\infty \leq n \max_{1 \leq i \leq n} \sum_{j=1}^n |h_j^{(i)}(\zeta_{ij})|_{\max} \cdot \|\mathbf{y} - \mathbf{x}\|_\infty. \quad (9)$$

For convenience, we construct  $\mathbf{X}_0 = \text{midrad}(\mathbf{x}_0, \omega)$  with  $\mathbf{x}_0$  as centre and  $\omega = 2\eta$  as radius.

Now we have  $|h_j^{(i)}(\zeta_{ij})|_{\max} \leq |h_j^{(i)}(\mathbf{X}_0)|_{\max}$ . So

$$\|\Delta \mathbf{J}\|_\infty \leq n \max_{1 \leq i \leq n} \sum_{j=1}^n |h_j^{(i)}(\mathbf{X}_0)|_{\max} \cdot \|\mathbf{y} - \mathbf{x}\|_\infty. \quad (10)$$

Therefore

$$K = n \max_{1 \leq i \leq n} \sum_{j=1}^n |h_j^{(i)}(\mathbf{X}_0)|_{\max} \quad (11)$$

is the Lipschitz constant w.r.t.  $\mathbf{J}$ .

Now we give an algorithm for computing initial intervals in Algorithm 1.

---

**Algorithm 1** init\_width

---

**Input:** Equation  $F$ ; Approximation  $x_0$ ; Number of variables  $n$

**Output:** Initial interval's radius  $r$

---

```

1: repeat
2:    $\mathbf{x}_0 = \mathbf{x}_0 - \mathbf{J}^{-1}(\mathbf{x}_0)\mathbf{F}(\mathbf{x}_0)$ ;
3:    $B = \|\mathbf{J}^{-1}(\mathbf{x}_0)\|_\infty$ ;  $\eta = \|\mathbf{J}^{-1}(\mathbf{x}_0)\mathbf{F}(\mathbf{x}_0)\|_\infty$ ;
4:    $\omega = 2\eta$ ;
5:    $\mathbf{X}_0 = \text{midrad}(\mathbf{x}_0, \omega)$ ;
6:    $K = 0$ ;
7:   for  $i = 1$  to  $n$  do
8:     Compute the Hessian matrix  $H_i = (h_1^{(i)}, h_2^{(i)}, \dots, h_n^{(i)})$  of  $\mathbf{F}$  on  $\mathbf{X}_0$ ;
9:     if  $\sum_{j=1}^n |h_j^{(i)}(\mathbf{X}_0)|_{\max} > K$  then
10:       $K = \sum_{j=1}^n |h_j^{(i)}(\mathbf{X}_0)|_{\max}$ ;
11:     end if
12:   end for
13:    $K = n \cdot K$ ;
14:    $h = BK\eta$ ;
15: until  $h \leq 1/2$ 
16: return  $r = \frac{1 - \sqrt{1 - 2h}}{h} \eta$ 

```

---



### 3.2 Empirical estimation

As so far, we have established a rigorous method to construct initial intervals. This method takes a complex approximate zero as input to obtain an initial box. But in practice we often find many approximations with “large” imaginary parts which strongly indicate that they are non-real. A natural question is

Can we detect these non-real roots without using interval arithmetic?

Let  $\mathbf{z}$  be an approximation of the real root  $\boldsymbol{\xi}$ . Because

$$\|\Re(\mathbf{z}) - \boldsymbol{\xi}\| \leq \|\mathbf{z} - \boldsymbol{\xi}\|,$$

then we can see the real part  $\Re(\mathbf{z})$  is also an approximation of this root and is even closer. So we can simply replace  $\mathbf{x}_0$  by  $\Re(\mathbf{x}_0)$  in Algorithm 1 to construct the initial box.

The other consideration is the efficiency of numerical computation. When we use Proposition 3, lots of interval matrix operations would be executed, which cost much time than the point operations. So if we can find an empirical estimate radius, which can be computed much faster, but is still valid for most of the equations, then that will be a good choice in practice.

We now give one such empirical estimation.

For  $\mathbf{F} = 0$ , let  $\mathbf{x}^*$  be an accurate root and  $\mathbf{x}_0$  be its approximation. Although the mean value theorem is not valid in complex space, the Taylor expand is still valid. And the polynomial systems considered here are all continuous, so we suppose the equation satisfies the mean value theorem approximately:

$$0 = \mathbf{F}(\mathbf{x}^*) \approx \mathbf{F}(\mathbf{x}_0) + \mathbf{J}(\boldsymbol{\xi})(\mathbf{x}^* - \mathbf{x}_0) \quad (12)$$

where  $\boldsymbol{\xi}$  is between  $\mathbf{x}^*$  and  $\mathbf{x}_0$ . So we have

$$\mathbf{x}^* - \mathbf{x}_0 \approx -\mathbf{J}^{-1}(\boldsymbol{\xi})\mathbf{F}(\mathbf{x}_0).$$

Let  $\mathbf{J}(\boldsymbol{\xi}) = \mathbf{J}(\mathbf{x}_0) + \Delta\mathbf{J}$ , then

$$\begin{aligned} \mathbf{J}(\boldsymbol{\xi}) &= \mathbf{J}(\mathbf{x}_0)(I + \mathbf{J}^{-1}(\mathbf{x}_0)\Delta\mathbf{J}), \\ \mathbf{J}^{-1}(\boldsymbol{\xi}) &= (I + \mathbf{J}^{-1}(\mathbf{x}_0)\Delta\mathbf{J})^{-1}\mathbf{J}^{-1}(\mathbf{x}_0). \end{aligned} \quad (13)$$

For  $\Delta\mathbf{J}$ , we can get an estimation similar to Formula (9):

$$\|\Delta\mathbf{J}\|_\infty \leq n \max_{1 \leq i \leq n} \sum_{j=1}^n |h_j^{(i)}(\zeta_{ij})|_{\max} \cdot \|\mathbf{x}^* - \mathbf{x}_0\|_\infty. \quad (14)$$

From our hypothesis,  $\mathbf{x}^*$  and  $\mathbf{x}_0$  are very close, so are  $\zeta_{ij}$  and  $\mathbf{x}_0$ . Thus, we approximate  $\mathbf{x}_0$  with  $\zeta_{ij}$ . Meanwhile, from  $\mathbf{x}_0$ , after a Newton iteration, we get  $\mathbf{x}_1 = \mathbf{x}_0 - \mathbf{J}^{-1}(\mathbf{x}_0)\mathbf{F}(\mathbf{x}_0)$ . Thus we may consider that the distance between  $\mathbf{x}^*$  and  $\mathbf{x}_0$  is more or less the same with that of  $\mathbf{x}_0$  and  $\mathbf{x}_1$ , so we replace  $\|\mathbf{x}^* - \mathbf{x}_0\|$  with  $\|\mathbf{x}_1 - \mathbf{x}_0\| = \|\mathbf{J}^{-1}(\mathbf{x}_0)\mathbf{F}(\mathbf{x}_0)\|$  for approximation.

So we get

$$\|\Delta \mathbf{J}\|_\infty \leq n \max_{1 \leq i \leq n} \sum_{j=1}^n |h_j^{(i)}(\mathbf{x}_0)|_{\max} \cdot \|\mathbf{J}^{-1}(\mathbf{x}_0) \mathbf{F}(\mathbf{x}_0)\|_\infty. \quad (15)$$

Let  $\lambda = \max_{1 \leq i \leq n} \sum_{j=1}^n |h_j^{(i)}(\mathbf{x}_0)|_{\max}$ , then

$$\|\mathbf{J}^{-1}(\mathbf{x}_0) \Delta \mathbf{J}\|_\infty \leq n\lambda \|\mathbf{J}^{-1}(\mathbf{x}_0)\|_\infty^2 \|\mathbf{F}(\mathbf{x}_0)\|_\infty.$$

Because  $\|\mathbf{F}(\mathbf{x}_0)\|_\infty \ll 1$ , the last formula is also far less than 1. So substitute that into Formula (13) we can get

$$\|\mathbf{J}^{-1}(\boldsymbol{\xi})\|_\infty \leq \frac{\|\mathbf{J}^{-1}(\mathbf{x}_0)\|_\infty}{1 - n\lambda \|\mathbf{J}^{-1}(\mathbf{x}_0)\|_\infty^2 \|\mathbf{F}(\mathbf{x}_0)\|_\infty}. \quad (16)$$

Finally we obtain the empirical estimation

$$\begin{aligned} \|\mathbf{x}^* - \mathbf{x}_0\|_\infty &\approx \|\mathbf{J}^{-1}(\boldsymbol{\xi}) \mathbf{F}(\mathbf{x}_0)\|_\infty \\ &\leq \frac{\|\mathbf{J}^{-1}(\mathbf{x}_0)\|_\infty \|\mathbf{F}(\mathbf{x}_0)\|_\infty}{1 - n\lambda \|\mathbf{J}^{-1}(\mathbf{x}_0)\|_\infty^2 \|\mathbf{F}(\mathbf{x}_0)\|_\infty}. \end{aligned} \quad (17)$$

Notice that the inequality (17) is only a non-rigorous estimation. All the computation in it are carried out in a point-wise way, so it is faster than Proposition 3. In the numerical experiments later we will see that this empirical estimate radius performs very well. So we can use it to detect those non-real roots rather than the interval arithmetic. We describe that in Algorithm 2.

---

**Algorithm 2** iscomplex

---

**Input:** Equation  $F$ ; Approximation  $\mathbf{z}$ ;

**Output:** **true** ( $\mathbf{z}$  must be non-real), or **false** ( $\mathbf{z}$  may be real).

1: Compute Formula (17), denote the result by  $r'$ ;

2: **if** any(  $\text{Im}(\mathbf{z}) > r'$  ) **then**

3:   **return true**; // not a real root, continue to judge others

4: **else**

5:   **return false**; // may be a real root, call interval arithmetic to verify

6: **end if**

---

In Algorithm 2, any() is a default function in Matlab, which returns true if there is non-zero component in a vector.

### 3.3 Krawczyk-Moore interval iteration

We now discuss about the real root verification with a given interval. In section 2.2, we have introduced the Krawczyk operator. With the properties in

Proposition 1, we can determine whether an interval contains a real root by the relationship of the original interval and the one after the Krawczyk iteration.

However, in practice, we can't expect the intervals to be entire inclusion or disjoint after just one iteration. Partly intersection is the most common cases that we encounter. Since the real root is still in the interval after the Krawczyk iteration, a normal method is to let  $\mathbf{X} \cap K(\mathbf{X})$  be the new iteration interval. So suppose  $\mathbf{X}^{(0)}$  is the initial interval, the iteration rule is  $\mathbf{X}^{(k+1)} = \mathbf{X}^{(k)} \cap K(\mathbf{X}^{(k)})$ , where  $K(\mathbf{X}^{(k)})$  is defined by Formula (4). This update rule can make sure that the size of  $\mathbf{X}^{(k)}$  is non-increasing. But a problem is once we encounter  $K(\mathbf{X}^{(k)}) \cap \mathbf{X}^{(k)} = \mathbf{X}^{(k)}$ , the iteration will be trapped into endless loop. So we have to divide  $\mathbf{X}^{(k)}$  if this happened.

Thus, we introduce a bisection function `divide()`. To ensure the convergence of our algorithm, we divide the longest dimension of an interval vector.

This strategy may not be the optimal choice when the system's dimension is high. Greedy method or optimization algorithm will be studied in future work.

We now give a formal description of `divide` function in Algorithm 3 and the Krawczyk-Moore iteration process in Algorithm 4.

---

**Algorithm 3** divide

---

**Input:** Interval vector  $\mathbf{X}$

**Output:**  $\mathbf{X}^{(1)}, \mathbf{X}^{(2)}$

- 1: Let  $\mathbf{X}_i$  be the coordinate with the largest width in  $\mathbf{X}$
  - 2:  $\mathbf{X}^{(1)} = \mathbf{X}; \mathbf{X}^{(2)} = \mathbf{X};$
  - 3:  $\mathbf{X}_i^{(1)} = [\inf(\mathbf{X}_i), \text{mid}(\mathbf{X}_i)];$
  - 4:  $\mathbf{X}_i^{(2)} = [\text{mid}(\mathbf{X}_i), \sup(\mathbf{X}_i)];$
  - 5: **return**  $\mathbf{X}^{(1)}, \mathbf{X}^{(2)}$
- 

### 3.4 Verification and refinement

After the Krawczyk iteration, we already have all the real root isolated intervals, but these are not the final results. Since we require an isolation of disjoint intervals, we have to check the possible overlaps.

On the other hand, some intervals may not be as small as required by users, so we can narrow them via bisection method until they match the requirement.

We discuss these details in this subsection.

**Remove the overlaps** There is a basic hypothesis: for non-singular systems, each root has an approximation, and from this approximation, the iteration will end up in its corresponding accurate root, not the other zeros. So we only have to remove the overlaps, and the number of real roots won't change.

However, we want to expand our algorithm into multi-roots cases. And in that situation, it is possible that two isolated intervals contain the same real

---

**Algorithm 4** Krawczyk

---

**Input:**  $F$ ; initial box  $\mathbf{X}$ ; isolation boxes  $real\_roots$ ; number of real roots  $nreal$

**Output:** symbol of whether there is a real root  $flag$ ;  $real\_roots$ ;  $nreal$

```
1:  $Y = \text{mid}(F'(\mathbf{X}))^{-1}$ ;  $\mathbf{X}_t = K(\mathbf{X})$ , where  $K(\mathbf{X})$  is define by Formula (4);
2: if  $\mathbf{X}_t \cap \mathbf{X} == \emptyset$  then
3:   return  $flag = \text{false}$ ;
4: end if
5: while not  $(\mathbf{X}_t \subseteq \mathbf{X})$  do
6:   if  $\mathbf{X}_t \cap \mathbf{X} == \mathbf{X}$  then
7:      $[\mathbf{X}^{(1)}, \mathbf{X}^{(2)}] = \text{divide}(\mathbf{X})$ ;
8:      $[f1, real\_roots, nreal] = \text{Krawczyk}(F, \mathbf{X}^{(1)}, real\_roots, nreal)$ ;
9:     if  $f1 == \text{false}$  then
10:       $[f2, real\_roots, nreal] = \text{Krawczyk}(F, \mathbf{X}^{(2)}, real\_roots, nreal)$ ;
11:     end if
12:     return  $f1$  or  $f2$ ;
13:   end if
14:    $\mathbf{X} = \mathbf{X}_t \cap \mathbf{X}$ ;
15:    $Y = (\text{mid}F'(\mathbf{X}))^{-1}$ .
16:    $\mathbf{X}_t = K(\mathbf{X})$ ;
17:   if  $\mathbf{X}_t \cap \mathbf{X} == \emptyset$  then
18:     return  $flag = \text{false}$ ;
19:   end if
20: end while
21:  $nreal = nreal + 1$ ;
22:  $real\_roots[nreal] = \mathbf{X}_t$ 
23: return  $flag = \text{true}, real\_roots, nreal$ ;
```

---

zero. So whether or not the overlap part contains a real root, our algorithm has its corresponding processes. See Algorithm 5 for details.

The function `Krawczyk()` in Algorithm 5 is a little bit different from that in Krawczyk-Moore iteration. In Krawczyk-Moore iteration, we have to store the information of isolated real root intervals, so the  $real\_roots$  and  $nreal$  are in the function arguments. However, we only need to know whether there is a real root here, so only the symbol variable  $flag$  is returned. The situation is the same in Algorithm 6.

**Narrow the width of intervals** As we said in the introduction, the real root isolation can be viewed as a kind of solving equations. And the width of the isolated intervals is just like the accuracy of solutions. Similar to the former algorithms, we can require the program to return an answer in specified range. The difference is the symbolic algorithm can get any precision they want in theory, but our floating point number calculation can't beat the machine precision. In fact, in the Matlab environment that we implement our algorithm, the results width won't be smaller than the system zero threshold<sup>1</sup>.

---

<sup>1</sup> In Matlab2008b that we do the experiments, the zero threshold is 2.2204e-016.

---

**Algorithm 5** disjoint\_process

---

**Input:** Isolated intervals  $real\_roots$ ; number of real roots  $nreal$ ;  $F$

**Output:** Checked isolated intervals  $real\_roots$ ;  $nreal$

```
1:  $k = 0$ ;
2: for  $i = 1$  to  $nreal$  do
3:    $X = real\_roots[i]$  ;  $new\_root = \text{true}$ ;
4:   for  $j = 1$  to  $k$  do
5:      $Y = real\_roots[j]$ ;
6:      $Z = X \cap Y$ ;
7:     if  $Z == \emptyset$  then
8:       continue;
9:     end if
10:     $flag = \text{Krawczyk}(F, Z)$ ;
11:    if  $flag == \text{true}$  then
12:       $new\_root = \text{false}$  ; break;
13:    else
14:       $X = X \setminus Z$ ;
15:       $real\_roots[j] = real\_roots[j] \setminus Z$ ;
16:    end if
17:  end for
18:  if  $new\_root == \text{true}$  then
19:     $k = k + 1$  ;  $real\_roots[k] = X$ 
20:  end if
21: end for
22: return  $real\_roots, nreal = k$ ;
```

---

We also use bisection to do the narrowing job. Since there is only one root in the interval, we only have to continue dividing and checking the half that contains that zero. Formal description is in Algorithm 6.

### 3.5 Algorithm description

Up to now, we have discussed all the parts of real root isolation algorithm in detail. We give the final main program in Algorithm 7.

## 4 Experiments

Now we apply our new method to some polynomial systems and do some comparison with some former algorithms.

All the experiments are undertaken in Matlab2008b, with Intlab [14] of Version 6. We use Hom4ps-2.0 [12] as our homotopy continuation tool to obtain initial approximate zeros. Since computation time will be listed below, the computer information is also given here: OS: Windows Vista, CPU: Inter®Core 2 Duo T6500 2.10GHz, Memory: 2G.

---

**Algorithm 6** narrowing

---

**Input:** Isolated intervals  $real\_roots$ ; Number of real roots  $nreal$ ;  $F$ ; Threshold  $\tau$

**Output:**  $real\_roots$

```
1: for  $i = 1$  to  $nreal$  do
2:    $\mathbf{X} = real\_roots[i]$ ;
3:   while any( $\text{rad}(\mathbf{X}) > \tau$ ) do
4:      $[\mathbf{Y}^{(1)}, \mathbf{Y}^{(2)}] = \text{divide}(\mathbf{X})$ ;
5:      $flag = \text{Krawczyk}(F, \mathbf{Y}^{(1)})$ ;
6:     if  $flag == \text{true}$  then
7:        $\mathbf{X} = \mathbf{Y}^{(1)}$ ;
8:     else
9:        $\mathbf{X} = \mathbf{Y}^{(2)}$ 
10:    end if
11:  end while
12:   $real\_roots[i] = \mathbf{X}$ ;
13: end for
14: return  $real\_roots$ 
```

---

#### 4.1 Demo example

We begin our illustration by a simple example.

**Example 1** Consider the real root isolation of the system below.

$$\begin{cases} x^3y^2 + x + 3 = 0 \\ 4yz^5 + 8x^2y^4z^4 - 1 = 0 \\ x + y + z - 1 = 0 \end{cases}$$

The homotopy program tells us this system has 28 complex zeros in total. And we get the following results after calling our `real_root_isolate` program.

```
intval =
[ -0.94561016957416, -0.94561016957415]
[ 1.55873837303161, 1.55873837303162]
[ 0.38687179654254, 0.38687179654255]
intval =
[ -1.18134319868123, -1.18134319868122]
[ -1.05029487815439, -1.05029487815438]
[ 3.23163807683560, 3.23163807683561]
intval =
[ -2.99999838968782, -2.99999838968781]
[ 0.00024421565895, 0.00024421565896]
[ 3.99975417402886, 3.99975417402887]
intval =
[ -0.79151164911096, -0.79151164911095]
[ 2.11038450699949, 2.11038450699950]
[ -0.31887285788855, -0.31887285788854]
```

The order of variables:

---

**Algorithm 7** real\_root\_isolate

---

**Input:** Equation  $F(\mathbf{x})$ ; number of variables  $n$ ; Threshold  $\tau$ ;

**Output:** Isolated intervals of  $F(\mathbf{x}) = 0$  and number of real roots  $nreal$

```
1:  $[complex\_roots, ncomplex] = \text{hom4ps}(F, n)$ ;  
2: Initialize  $real\_roots$  to be empty;  $nreal = 0$ ;  
3: for  $i = 1$  to  $ncomplex$  do  
4:    $z = complex\_roots[i]$ ;  
5:   if  $\text{iscomplex}(F, z)$  then  
6:     continue;  
7:   end if  
8:    $r = \text{init\_width}[F, z, n]$ ;  
9:    $X_0 = \text{midrad}(\Re(z), r)$ ;  
10:   $[flag, real\_roots, nreal] = \text{Krawczyk}(F, X_0, real\_roots, nreal)$ ;  
11: end for  
12:  $[real\_roots, nreal] = \text{disjoint\_process}(real\_roots, nreal, F)$ ;  
13:  $real\_roots = \text{narrowing}(real\_roots, nreal, F, \tau)$ ;  
14: return  $real\_roots, nreal$ ;
```

---

'x'

'y'

'z'

The number of real roots: 4

We verify the answers above with the DISCOVERER [22] package under Maple, which also return 4 isolated real roots. Here we show its output in floating point number format, i.e.

$[[-2.999998391, -2.999998389], [0.2442132427\text{e-}3, 0.2442180230\text{e-}3], [3.999754090, 3.999754249]]$ ,  
 $[[-1.181343199, -1.181343199], [-1.050294975, -1.050294818], [3.231637836, 3.231638372]]$ ,  
 $[[-.9456101805, -.9456101656], [1.558738033, 1.558738728], [3.868716359, .3868719935]]$ ,  
 $[[-.7915116549, -.7915116400], [2.110384024, 2.110385000], [-.3188729882, -.3188727498]]$ .

And we can see the answers perfectly match the ones of our program.

We list some information during the calculation of our algorithm here for reference. Only the 4 real ones are given, and the other non-real ones are all detected by our empirical estimation method. We mention that all the imaginary parts of complex roots are significant larger than the initial radius of our algorithm in order of magnitude in this example.

	root1	root2	root3	root4
$B$	1.060227	1.192159	2.000864	0.874354
$K$	14.941946	7.198937e+003	4.095991e+003	16.988990
$\eta$	4.260422e-016	4.20807e-016	8.882333e-016	5.764449e-016
$h$	2.024791e-014	1.083446e-011	2.183861e-011	2.568823e-014
estimate-rad	4.274976e-016	4.208067e-016	8.882344e-016	5.779921e-016
empirical-rad	1.015249e-015	1.29164e-012	2.156138e-015	1.559270e-015

**Table 1.** Key quantities comparison

We give some remarks on table 1. In the first row, *root1* to *root4* are refer to the 4 real roots mentioned above respectively. And  $B, K, \eta, h$  are exactly the same as they are defined in algorithm 1. The *estimate-rad* are the radius obtained via algorithm 1, while the *empirical-rad* are refer to the ones calculated by Formula (17).

We say a little more words about the *empirical-rad*. Firstly, although the empirical ones are basically larger than the rigorous error radius, they are still small enough, which hardly have any influence on the efficiency of interval iteration. We will see this in the comparison experiments later. But avoiding of interval matrix computation is very helpful to the algorithm. Secondly, the radius obtained from algorithm 1 are so small that they are even comparable to the zero threshold of Matlab system<sup>1</sup>. And this could bring some uncertainty of floating point operation to our algorithm, such as misjudgement of interval inclusion in Intlab, etc. So we intend to use empirical estimation bound in next experiments.

For *cyclic6* (see Appendix A), the classic symbolic algorithm can do nothing due to the difficulty of triangularization. Meanwhile, we can easily get the 24 isolated real roots intervals with our `real_root_isolate` program.

## 4.2 Comparison experiment

Many benchmarks have been checked with our `real_root_isolate` program. Since the time complexity of both triangularization and homotopy continuation are difficult to be analyzed, we mainly focus on the isolation results and the program execution time.

We investigate over 130 benchmarks provided by Hom4ps [1], among which about 40 equations are non-singular systems. We apply our program to these equations and all the experiments receive the right answers. Here we list a few of them (see Appendix A for details).

The column *real roots* in Table 2 tells the number of intervals that our program isolated. Compared with the results of DISCOVERER, the new algorithm indeed works out all equations that are beyond the capability of classic symbolic algorithm. Moreover, the last column show that our empirical estimate method detects all the non-real roots successfully.

Table 3 gives the comparison of program execution time. The total time is not equal to the sum of homotopy time and interval iteration time because we only count the CPU time, and there are other tasks such as I/O, format transform, etc.

Table 3 also shows that interval iterations consume more time than homotopy continuation. The reason is complicated and we enumerate some here:

1. The homotopy focus only on floating point number, while Krawczyk cares about intervals;

---

<sup>1</sup> As mentioned before, the zero threshold in Matlab2008b is 2.2204e-016, which is almost the same order of magnitude of those radiuses.



Equation	total roots	real roots	DISCOVERER	complex roots detected
barry	20	2	2	18
cyclic5	70	10	10	60
cyclic6	156	24	N/A	132
des18_3	46	6	N/A	40
eco7	32	8	8	24
eco8	64	8	N/A	56
geneig	10	10	N/A	0
kinema	40	8	N/A	32
reimer4	36	8	8	28
reimer5	144	24	N/A	120
virasoro	256	224	N/A	32

**Table 2.** Real root isolation results comparison

Equations	Total time	Homotopy time	Interval time	DISCOVERER
barry	0.421203	0.093601	0.327602	0.063
cyclic5	2.948419	0.218401	2.652017	0.624
cyclic6	9.984064	0.639604	9.063658	N/A
des18_3	4.180827	0.702004	3.385222	N/A
eco7	2.371215	0.265202	2.012413	15.881
eco8	3.946825	0.499203	3.354022	N/A
geneig	4.243227	0.249602	3.868825	N/A
kinema	3.946825	1.014006	2.808018	N/A
reimer4	2.480416	0.374402	2.059213	24.711
reimer5	12.963683	3.073220	9.578461	N/A
virasoro	137.124879	4.570829	109.996305	N/A

**Table 3.** Execution time comparison, unit:s

2. Hom4ps-2.0 is a software compiled from language C, which is much more efficient than the tool that we use to implement our algorithm, say Matlab.
3. The interval iteration time increases as roots number grows since we examine the approximate zeros one by one. So the parallel computation of homotopy is much faster.

We believe that with efficient language such as C/C++, and parallel computation, the implementation of our algorithm will be much faster.

In order to verify our idea and see whether parallelization could help, we go into every approximate zero's iteration process. Some critical data are recorded in table 4. The *avg. rad. of ans* is the average radius of the final isolated intervals, while the *avg. rad. of init.* indicates the average radius of the initial intervals. The average time of each zero's interval iteration is shown in column *avg. time of iteration* along with the max interval iteration time in *max time of iter.* We think the consumption for each zero's process is acceptable.

From table 4 we can see that the initial interval radiuses are extremely small, which leads to a nice process time for each iteration. We point out that almost

Equation	avg.rad.of ans	avg.rad.of init.	avg.time of iter	max time of iter
barry	3.552714e-015	1.377800e-014	0.054600	0.062400
cyclic5	1.614703e-009	7.142857e-007	0.113881	0.140401
cyclic6	4.440892e-016	2.137195e-015	0.183951	0.234002
des18_3	3.768247e-007	9.737288e-007	0.241802	0.296402
eco7	1.998401e-015	1.483754e-013	0.122851	0.156001
eco8	2.109424e-015	3.283379e-013	0.183301	0.218401
geneig	2.664535e-016	5.721530e-014	0.315122	0.436803
kinema	1.998401e-015	6.784427e-011	0.157951	0.218401
reimer4	1.110223e-016	1.258465e-014	0.122851	0.156001
reimer5	1.110223e-016	4.754080e-014	0.195001	0.421203
virasoro	9.472120e-009	2.265625e-006	0.387844	0.624004

**Table 4.** Detail data for each iteration, unit:s

all real root checks are done by just *one* Krawczyk iteration, and hardly any overlap is found after all the Krawczyk iteration processes due to the small initial intervals that we give. All of these save significant execute time of our program.

## 5 Conclusion

For the non-singular polynomial systems with variables' number equals equations' number, this paper presents a new algorithm for real root isolation based on hybrid computation. The algorithm first applies homotopy continuation to obtain all the initial approximate zeros of the system. For each approximate zero, an initial interval which contains the corresponding accurate root is constructed. Then Krawczyk operator is called to verify all the initial intervals so as to get all the real root isolated boxes. Some necessary check and refinement work are done after that to ensure the boxes are pairwise disjoint and meet width requirement.

In the construction of initial intervals, we give a rigorous radius error bound based on the corollary of Kantorovich theorem. Some constructive algorithms are presented for both real and complex approximate zeros. Meanwhile, we introduce an empirical estimate radius, which has a nice performance in numerical experiments.

In the modification and implementation of Krawczyk iteration algorithm, some problems of interval arithmetic are also discussed in this paper.

At last we utilize some existing tools to implement our algorithm under Matlab environment. Many benchmarks have been checked along with comparison and analysis.

We also mention some possible future work here.

The construction of initial intervals is still too complicated and further optimization shall be studied. Also the empirical estimation with more efficiency and accuracy is a considerable question. The divide strategy in Krawczyk iteration could also be improved, which may be helpful in the high dimension cases.

In the aspect of implementation, replacing the Matlab implementation with C/C++ codes may improve the performance of our program. Parallel computation can solve another bottleneck of our problem. And for some small systems, or equations with special property, the classic symbolic algorithm could be even faster. So the tradeoff of symbolic and numerical computation is also an interesting direction.

## Acknowledgements

The work is partly supported by the ANR-NSFC project EXACTA (ANR-09-BLAN-0371-01/60911130369), NSFC project (11001040) and the project SYSKF1207 from State Key Laboratory of Computer Science, Institute of Software, Chinese Academy of Sciences. The authors especially thank professor Dongming Wang for the early discussion on this topic in 2010. And also thank professor T. Y. Li for his helpful suggestions and his team's work on Hom4ps2-Matlab interface.

## References

1. Benchmarks: [http://hom4ps.math.msu.edu/HOM4PS\\_soft\\_files/equations.zip](http://hom4ps.math.msu.edu/HOM4PS_soft_files/equations.zip).
2. Blum, L. and Cucker, F. and Shub, M. and Smale, S.: Complexity and Real Computation. Springer, 1997.
3. Boulier, F. and Chen C.B. and Lemaire, F. and Moreno Maza., M.: Real Root Isolation of Regular Chains. In: Proceedings of the 2009 Asian Symposium on Computer Mathematics (ASCM 2009), 1–15, 2009.
4. Cheng J.S. and Gao X.S. and Guo L.L.: Root isolation of zero-dimensional polynomial systems with linear univariate representation. *Journal of Symbolic Computation*, 47(7):843–858, 2012.
5. Cheng J.S., Gao X.S. and Yap C.K.: Complete Numerical Isolation of Real Zeros in General Triangular Systems. In: Proc. 2007 Internat. Symp. Symbolic Algebraic Comput. (ISSAC'07), 92–99, 2007.
6. Collins, G.E. and Loos, R.: Real zeros of polynomials. In: Computer Algebra: Symbolic and Algebraic Computation, (Buchberger, B. and Collins, G.E. and Loos, R. eds.), Springer-Verlag, 83–94, 1982.
7. Dayton, B. and Li T.Y. and Zeng Z.G.: Multiple zeros of nonlinear systems, *Math. Comp.*, 80:2143–2168, 2011.
8. Gragg, G.W. and Tapia, R.A.: Optimal Error Bounds for the Newton-Kantorovich Theorem. *SIAM Journal on Numerical Analysis*, 11(1):10–13, 1974.
9. Leykin, A. and Verschelde, J. and Zhao A.L.: Newton's method with deflation for isolated singularities of polynomial systems. *Theoretical Computer Science*, 359:111–122, 2006.
10. Leykin, A. and Verschelde, J. and Zhao A.L.: Higher-order Deflation for Polynomial Systems With Isolated Singular Solutions. In: Algorithms in Algebraic Geometry (Dickstein, A. and Schreyer, F. and Sommese, A., eds.), Springer, 2008, 79–97.
11. Li T.Y.: Numerical solution of multivariate polynomial systems by homotopy continuation methods. *Acta Numerica*, 6:399–436, 1997.
12. Li T.Y.: HOM4PS-2.0. [http://hom4ps.math.msu.edu/HOM4PS\\_soft.htm](http://hom4ps.math.msu.edu/HOM4PS_soft.htm), 2008.

13. Rouillier F.: Solving Zero-Dimensional Systems Through the Rational Univariate Representation. *Applicable Algebra in Engineering, Communication and Computing*, 9:433–461, 1999.
14. Rump, S.M.: INTLAB - INTeRval LABoratory. In: *Developments in Reliable Computing* (Tibor Csendes ed.), 77–104, Kluwer Academic Publishers, 1999. <http://www.ti3.tu-harburg.de/rump/>
15. Rump, S.M.: Verification methods: Rigorous results using floating-point arithmetic. *Acta Numerica*, 19:287–449, 2010.
16. Shen F.: The Real Roots Isolation of Polynomial System Based on Hybrid Computation. Master degree thesis, Peking University, April, 2012.
17. Sommese, A. and Wampler, C.: *The Numerical Solution of Systems of Polynomials: Arising in Engineering and Science*. World Scientific, 2005.
18. Verschelde, J.: PHCpack. <http://homepages.math.uic.edu/~jan/PHCpack/phcpack.html>, 1999.
19. Wampler, C.: HomLab. <http://nd.edu/~cwample1/HomLab/main.html>, 2005.
20. Wang D.R. and Zhang L.S. and Deng N.Y.: *Interval algorithms for nonlinear equations*. Shanghai science and technology press, 1996.
21. Wu X.L. and Zhi L.H.: Computing the multiplicity structure from geometric involutive form. In: *Proc. 2008 Internat. Symp. Symbolic Algebraic Comput. (ISSAC'08)*, 325–332, 2008.
22. Xia B.C.: DISCOVERER: A tool for solving semi-algebraic systems. *ACM Communications in Computer Algebra*, 41(3):102–103, 2009.
23. Xia B. and Zhang T.: Real Solution Isolation Using Interval Arithmetic. *Computers and Mathematics with Applications*, 52:853–860, 2006.
24. Yang L. and Xia B.: An algorithm for isolating the real solutions of semi-algebraic systems. *Journal of Symbolic Computation*, 34:461–477, 2002.
25. Zhang T.: Isolating Real Roots of Nonlinear Polynomial. Master degree thesis, Peking University, 2004.
26. Zhang T. and Xiao R. and Xia B.C.: Real Solution Isolation Based on Interval Krawczyk Operator. In: *Proc. the 7th Asian Symposium on Computer Mathematics (ASCM 2005)* (Pae, Sung-il and Park, H. eds.), 235–237, 2005.

## A Benchmarks

1. barry: Number of variables:3,Number of equations:3,Max degree:5

$$\begin{aligned}
 -x^5 + y^5 - 3y - 1 &= 0 \\
 5y^4 - 3 &= 0 \\
 -20x + y - z &= 0
 \end{aligned}$$

2. cyclic5: Number of variables:5,Number of equations:5,Max degree:5

$$\begin{aligned}
 x_1 + x_2 + x_3 + x_4 + x_5 &= 0 \\
 x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_1x_5 &= 0 \\
 x_1x_2x_3 + x_2x_3x_4 + x_3x_4x_5 + x_4x_5x_1 + x_5x_1x_2 &= 0 \\
 x_1x_2x_3x_4 + x_2x_3x_4x_5 + x_3x_4x_5x_1 + x_4x_5x_1x_2 + x_5x_1x_2x_3 &= 0 \\
 x_1x_2x_3x_4x_5 - 1 &= 0
 \end{aligned}$$

3. cyclic6: Number of variables:6,Number of equations:6,Max degree:6

$$\begin{aligned}
x_0 + x_1 + x_2 + x_3 + x_4 + x_5 &= 0 \\
x_0x_1 + x_1x_2 + x_2x_3 + x_3x_4 + x_4x_5 + x_5x_0 &= 0 \\
x_0x_1x_2 + x_1x_2x_3 + x_2x_3x_4 + x_3x_4x_5 + x_4x_5x_0 + x_5x_0x_1 &= 0 \\
x_0x_1x_2x_3 + x_1x_2x_3x_4 + x_2x_3x_4x_5 + x_3x_4x_5x_0 \\
&+ x_4x_5x_0x_1 + x_5x_0x_1x_2 = 0 \\
x_0x_1x_2x_3x_4 + x_1x_2x_3x_4x_5 + x_2x_3x_4x_5x_0 + x_3x_4x_5x_0x_1 \\
&+ x_4x_5x_0x_1x_2 + x_5x_0x_1x_2x_3 = 0 \\
x_0x_1x_2x_3x_4x_5 - 1 &= 0
\end{aligned}$$

4. des18\_3: Number of variables:8,Number of equations:8,Max degree:3

$$\begin{aligned}
15a_{33}a_{10}a_{21} - 162a_{10}^2a_{22} - 312a_{10}a_{20} + 24a_{10}a_{30} + 27a_{31}a_{21} \\
+ 24a_{32}a_{20} + 18a_{22}a_{10}a_{32} + 30a_{22}a_{30} + 84a_{31}a_{10} &= 0 \\
28a_{22}a_{10}a_{33} + 192a_{30} + 128a_{32}a_{10} + 36a_{31}a_{20} + 36a_{33}a_{20} \\
- 300a_{10}a_{21} + 40a_{32}a_{21} - 648a_{10}^2 + 44a_{22}a_{31} &= 0 \\
180a_{33}a_{10} - 284a_{22}a_{10} - 162a_{10}^2 + 60a_{22}a_{32} + 50a_{32}a_{10} \\
+ 70a_{30} + 55a_{33}a_{21} + 260a_{31} - 112a_{20} &= 0 \\
6a_{33}a_{10}a_{20} + 10a_{22}a_{10}a_{31} + 8a_{32}a_{10}a_{21} - 162a_{10}^2a_{21} \\
+ 16a_{21}a_{30} + 14a_{31}a_{20} + 48a_{10}a_{30} &= 0 \\
4a_{22}a_{10}a_{30} + 2a_{32}a_{10}a_{20} + 6a_{20}a_{30} - 162a_{10}^2a_{20} + 3a_{31}a_{21}a_{10} &= 0 \\
66a_{33}a_{10} + 336a_{32} + 90a_{31} + 78a_{22}a_{33} - 1056a_{10} - 90a_{21} &= 0 \\
- 240a_{10} + 420a_{33} - 64a_{22} + 112a_{32} &= 0 \\
136a_{33} - 136 &= 0
\end{aligned}$$

5. eco7: Number of variables:7,Number of equations:7,Max degree:3

$$\begin{aligned}
x_7x_1 + x_7x_1x_2 + x_7x_2x_3 + x_7x_3x_4 + x_7x_4x_5 + x_7x_5x_6 - 1 &= 0 \\
x_7x_2 + x_7x_1x_3 + x_7x_2x_4 + x_7x_3x_5 + x_7x_6x_4 - 2 &= 0 \\
x_7x_3 + x_7x_1x_4 + x_7x_2x_5 + x_7x_6x_3 - 3 &= 0 \\
x_7x_4 + x_7x_1x_5 + x_7x_2x_6 - 4 &= 0 \\
x_7x_5 + x_7x_1x_6 - 5 &= 0 \\
x_6x_7 - 6 &= 0 \\
x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + 1 &= 0
\end{aligned}$$

6. eco8: Number of variables:8,Number of equations:8,Max degree:3

$$\begin{aligned}
& x_1x_8 + x_8x_1x_2 + x_8x_2x_3 + x_8x_3x_4 + x_8x_4x_5 \\
& \quad + x_8x_5x_6 + x_8x_6x_7 - 1 = 0 \\
& x_2x_8 + x_8x_1x_3 + x_8x_2x_4 + x_8x_3x_5 + x_8x_6x_4 + x_8x_7x_5 - 2 = 0 \\
& x_8x_3 + x_8x_1x_4 + x_8x_2x_5 + x_8x_6x_3 + x_8x_7x_4 - 3 = 0 \\
& x_8x_4 + x_8x_1x_5 + x_8x_2x_6 + x_8x_7x_3 - 4 = 0 \\
& x_8x_5 + x_8x_1x_6 + x_8x_7x_2 - 5 = 0 \\
& x_8x_6 + x_8x_7x_1 - 6 = 0 \\
& x_7x_8 - 7 = 0 \\
& x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + 1 = 0
\end{aligned}$$

7. geneig: Number of variables:6,Number of equations:6,Max degree:3

$$\begin{aligned}
& -10x_1x_6^2 + 2x_2x_6^2 - x_3x_6^2 + x_4x_6^2 + 3x_5x_6^2 + x_1x_6 + 2x_2x_6 \\
& \quad + x_3x_6 + 2x_4x_6 + x_5x_6 + 10x_1 + 2x_2 - x_3 + 2x_4 - 2x_5 = 0 \\
& 2x_1x_6^2 - 11x_2x_6^2 + 2x_3x_6^2 - 2x_4x_6^2 + x_5x_6^2 + 2x_1x_6 + x_2x_6 \\
& \quad + 2x_3x_6 + x_4x_6 + 3x_5x_6 + 2x_1 + 9x_2 + 3x_3 - x_4 - 2x_5 = 0 \\
& -x_1x_6^2 + 2x_2x_6^2 - 12x_3x_6^2 - x_4x_6^2 + x_5x_6^2 + x_1x_6 + 2x_2x_6 \\
& \quad - 2x_4x_6 - 2x_5x_6 - x_1 + 3x_2 + 10x_3 + 2x_4 - x_5 = 0 \\
& x_1x_6^2 - 2x_2x_6^2 - x_3x_6^2 - 10x_4x_6^2 + 2x_5x_6^2 + 2x_1x_6 + x_2x_6 \\
& \quad - 2x_3x_6 + 2x_4x_6 + 3x_5x_6 + 2x_1 - x_2 + 2x_3 + 12x_4 + x_5 = 0 \\
& 3x_1x_6^2 + x_2x_6^2 + x_3x_6^2 + 2x_4x_6^2 - 11x_5x_6^2 + x_1x_6 + 3x_2x_6 \\
& \quad - 2x_3x_6 + 3x_4x_6 + 3x_5x_6 - 2x_1 - 2x_2 - x_3 + x_4 + 10x_5 = 0 \\
& x_1 + x_2 + x_3 + x_4 + x_5 - 1 = 0
\end{aligned}$$

8. kinema: Number of variables:9,Number of equations:9,Max degree:2

$$\begin{aligned}
& z_1^2 + z_2^2 + z_3^2 - 12z_1 - 68 = 0 \\
& z_4^2 + z_5^2 + z_6^2 - 12z_5 - 68 = 0 \\
& z_7^2 + z_8^2 + z_9^2 - 24z_8 - 12z_9 + 100 = 0 \\
& z_1z_4 + z_2z_5 + z_3z_6 - 6z_1 - 6z_5 - 52 = 0 \\
& z_1z_7 + z_2z_8 + z_3z_9 - 6z_1 - 12z_8 - 6z_9 + 64 = 0 \\
& z_4z_7 + z_5z_8 + z_6z_9 - 6z_5 - 12z_8 - 6z_9 + 32 = 0 \\
& 2z_2 + 2z_3 - z_4 - z_5 - 2z_6 - z_7 - z_9 + 18 = 0 \\
& z_1 + z_2 + 2z_3 + 2z_4 + 2z_6 - 2z_7 + z_8 - z_9 - 38 = 0 \\
& z_1 + z_3 - 2z_4 + z_5 - z_6 + 2z_7 - 2z_8 + 8 = 0
\end{aligned}$$

9. reimer4: Number of variables:4,Number of equations:4,Max degree:5

$$\begin{aligned}2x_1^2 - 2x_2^2 + 2x_3^2 - 2x_4^2 - 1 &= 0 \\2x_1^3 - 2x_2^3 + 2x_3^3 - 2x_4^3 - 1 &= 0 \\2x_1^4 - 2x_2^4 + 2x_3^4 - 2x_4^4 - 1 &= 0 \\2x_1^5 - 2x_2^5 + 2x_3^5 - 2x_4^5 - 1 &= 0\end{aligned}$$

10. reimer5: Number of variables:5,Number of equations:5,Max degree:6

$$\begin{aligned}2x_1^2 - 2x_2^2 + 2x_3^2 - 2x_4^2 + 2x_5^2 - 1 &= 0 \\2x_1^3 - 2x_2^3 + 2x_3^3 - 2x_4^3 + 2x_5^3 - 1 &= 0 \\2x_1^4 - 2x_2^4 + 2x_3^4 - 2x_4^4 + 2x_5^4 - 1 &= 0 \\2x_1^5 - 2x_2^5 + 2x_3^5 - 2x_4^5 + 2x_5^5 - 1 &= 0 \\2x_1^6 - 2x_2^6 + 2x_3^6 - 2x_4^6 + 2x_5^6 - 1 &= 0\end{aligned}$$

11. virasoro: Number of variables:8,Number of equations:8,Max degree:2

$$\begin{aligned}2x_1x_4 - 2x_1x_7 + 2x_2x_4 - 2x_2x_6 + 2x_3x_4 - 2x_3x_5 + 8x_4^2 \\+ 8x_4x_5 + 2x_4x_6 + 2x_4x_7 + 6x_4x_8 - 6x_5x_8 - x_4 &= 0 \\2x_1x_5 - 2x_1x_6 + 2x_2x_5 - 2x_2x_7 - 2x_3x_4 + 2x_3x_5 + 8x_4x_5 \\- 6x_4x_8 + 8x_5^2 + 2x_5x_6 + 2x_5x_7 + 6x_5x_8 - x_5 &= 0 \\-2x_1x_5 + 2x_1x_6 - 2x_2x_4 + 2x_2x_6 + 2x_3x_6 - 2x_3x_7 + 2x_4x_6 \\+ 2x_5x_6 + 8x_6^2 + 8x_6x_7 + 6x_6x_8 - 6x_7x_8 - x_6 &= 0 \\-2x_1x_4 + 2x_1x_7 - 2x_2x_5 + 2x_2x_7 - 2x_3x_6 + 2x_3x_7 + 2x_4x_7 \\+ 2x_5x_7 + 8x_6x_7 - 6x_6x_8 + 8x_7^2 + 6x_7x_8 - x_7 &= 0 \\8x_1^2 + 8x_1x_2 + 8x_1x_3 + 2x_1x_4 + 2x_1x_5 + 2x_1x_6 \\+ 2x_1x_7 - 8x_2x_3 - 2x_4x_7 - 2x_5x_6 - x_1 &= 0 \\8x_1x_2 - 8x_1x_3 + 8x_2^2 + 8x_2x_3 + 2x_2x_4 + 2x_2x_5 \\+ 2x_2x_6 + 2x_2x_7 - 2x_4x_6 - 2x_5x_7 - x_2 &= 0 \\-8x_1x_2 + 8x_1x_3 + 8x_2x_3 + 8x_3^2 + 2x_3x_4 + 2x_3x_5 \\+ 2x_3x_6 + 2x_3x_7 - 2x_4x_5 - 2x_6x_7 - x_3 &= 0 \\-6x_4x_5 + 6x_4x_8 + 6x_5x_8 - 6x_6x_7 + 6x_6x_8 + 6x_7x_8 + 8x_8^2 - x_8 &= 0\end{aligned}$$